Coin tossing sequences

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• Toss a coin repeatedly until we get a particular sequence.
• e.g. HTT

![9 tosses]

• How many tosses on average?
• Is it the same for all sequences?
How many tosses on average to get HTT or HTH?

Options:

A. HTT takes longer

B. HTH takes longer

C. Both the same
Target: HTT
Probability = 1/8
Average wait = 8
Target: HTH

Overlaps don’t get counted

Probability = 1/8

Average wait > 8
Calculating the average wait

Target: HTH

Stake 1 1 1 1 1 1 1 1 1 1 1 1
Payout 0 0 0 0 0 0 0 8 0 2 8 10

- Bet £1 on each coin being start of chosen sequence
- In a fair game, payout for matching all 3 is £8
- Also payout £4 for matching 2, £2 for matching 1
- For HTH, payout = £10
- For fair game, average stake = payout

∴ Average number of tosses = 10 for HTH
Calculating the average wait

Target: HTH

| Stake (£) | 1 1 1 1 1 1 1 1 1 1 1 |
| Payout (£) | 0 0 0 0 0 8 0 2 10 |

- Bet £1 on each coin being start of chosen sequence
- In a fair game, payout for matching all 3 is £8
- Also payout £4 for matching 2, £2 for matching 1
- For HTH, payout = £10
- For fair game, average stake = payout

∴ Average number of tosses = 10 for HTH
### Average wait

<table>
<thead>
<tr>
<th>Sequence</th>
<th>Average wait</th>
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<tbody>
<tr>
<td>H</td>
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<tr>
<td>T</td>
<td>2</td>
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<table>
<thead>
<tr>
<th>Sequence</th>
<th>Average wait</th>
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</thead>
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<td>HT</td>
<td>4</td>
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<tr>
<td>TH</td>
<td>4</td>
</tr>
<tr>
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<table>
<thead>
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<th>Sequence</th>
<th>Average wait</th>
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<tr>
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</table>

**Sequence length $n$**

- Minimum wait $= 2^n$
- Maximum wait $= 2^{n+1} - 2$
- Longer sequence always has longer wait
Two sequences

• Which is more likely to occur first?
• e.g. HTT, HTH

• Penney’s game

# Probability of red sequence preceding blue

## $n=3$

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</tr>
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</table>
Probability of red sequence preceding blue

$n=3$
Which sequence is more likely to occur first

\[
n=3
\]

Shorter wait always at least 50% probability of preceding longer one

Average wait
Which sequence is more likely to occur first

\[ n=4 \]
Which sequence is more likely to occur first

$n=9$
Which sequence is more likely to occur first

\[ n=4 \]
Which sequence has the shorter waiting time

\( n=4 \)
Which sequence is more likely to occur first

Longer average wait is more likely to precede shorter one!

$n=4$
Shorter wait can precede longer wait

Ignoring transposition of H,T, distinct cases are:

• HTHH (18) beats HHTT (16) with $p=\frac{4}{7} \approx 0.57$
• HTHH (18) beats THHH (16) with $p=\frac{7}{12} \approx 0.58$
• THHT (18) beats HHTT (16) with $p=\frac{7}{12} \approx 0.58$
• THTH (20) beats HTHH (18) with $p=\frac{9}{14} \approx 0.64$

THTH has an almost 2/3 probability of preceding HTHH, yet takes longer to occur on average!
References

- Martin Gardner, Time Travel and other Mathematical Bewilderments, 60-66
- [https://plus.maths.org/content/os/issue55/features/nishiyama/index](https://plus.maths.org/content/os/issue55/features/nishiyama/index)
- Same article, but also including proof based on average wait time: [http://www.i-repository.net/il/user_contents/02/G0000031Repository/repository/keidaironshu_063_004_269-276.pdf](http://www.i-repository.net/il/user_contents/02/G0000031Repository/repository/keidaironshu_063_004_269-276.pdf)
- Theorems X and Y in this article can be seen to be true based on the gambling approach shown in the following
- Penney’s game [https://en.wikipedia.org/wiki/Penney%27s_game](https://en.wikipedia.org/wiki/Penney%27s_game)