

Numerical frieze patterns

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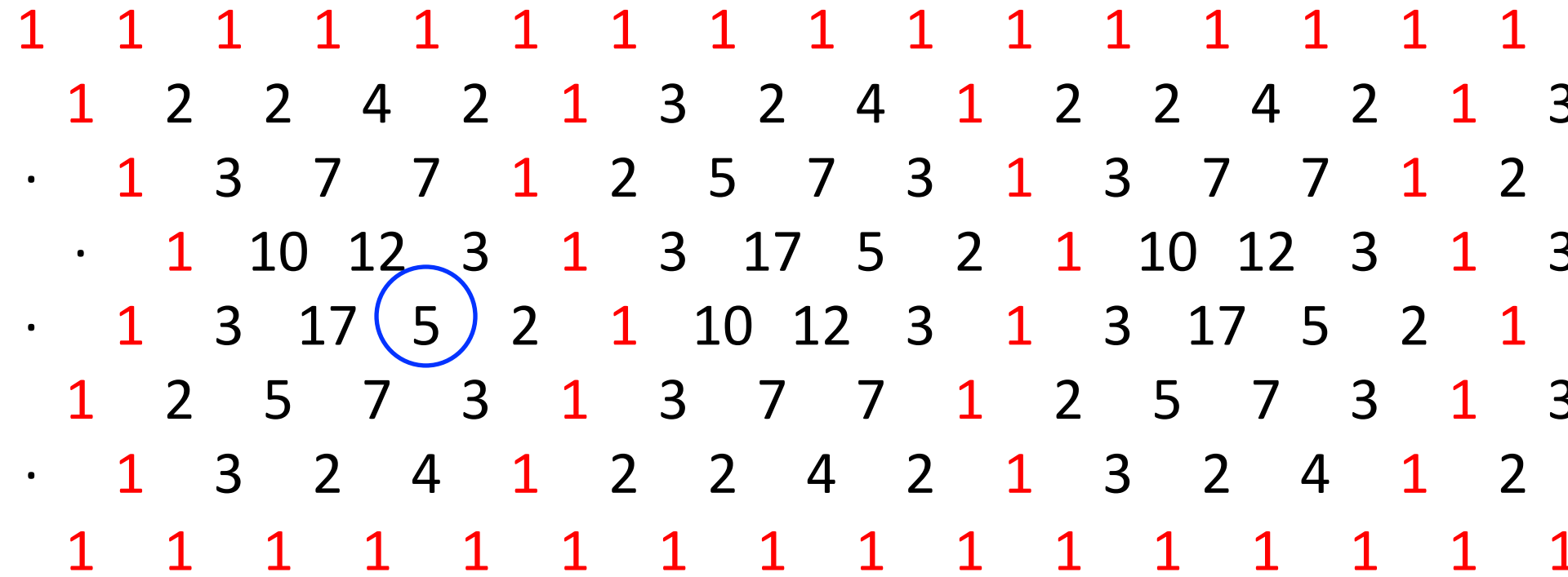
@MB_Whitworth

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
4	1	2	2	3	2	4	1	2	2	4	4	1	2	2	3
3	1	3	5	5	7	3	1	3	7	15	3	1	3	5	5
11	2	1	7	8	17	5	2	1	10	26	11	2	1	7	8
7	1	2	11	27	12	3	1	3	37	19	7	1	2	11	27
12	3	1	3	37	19	7	1	2	11	27	12	3	1	3	37
5	2	1	10	26	11	2	1	7	8	17	5	2	1	10	26
7	3	1	3	7	15	3	1	3	5	5	7	3	1	3	7
4	1	2	2	4	4	1	2	2	3	2	4	1	2	2	4
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

- What is the pattern?

$$\begin{matrix} & a & \\ b & & c \\ & d & \end{matrix}$$

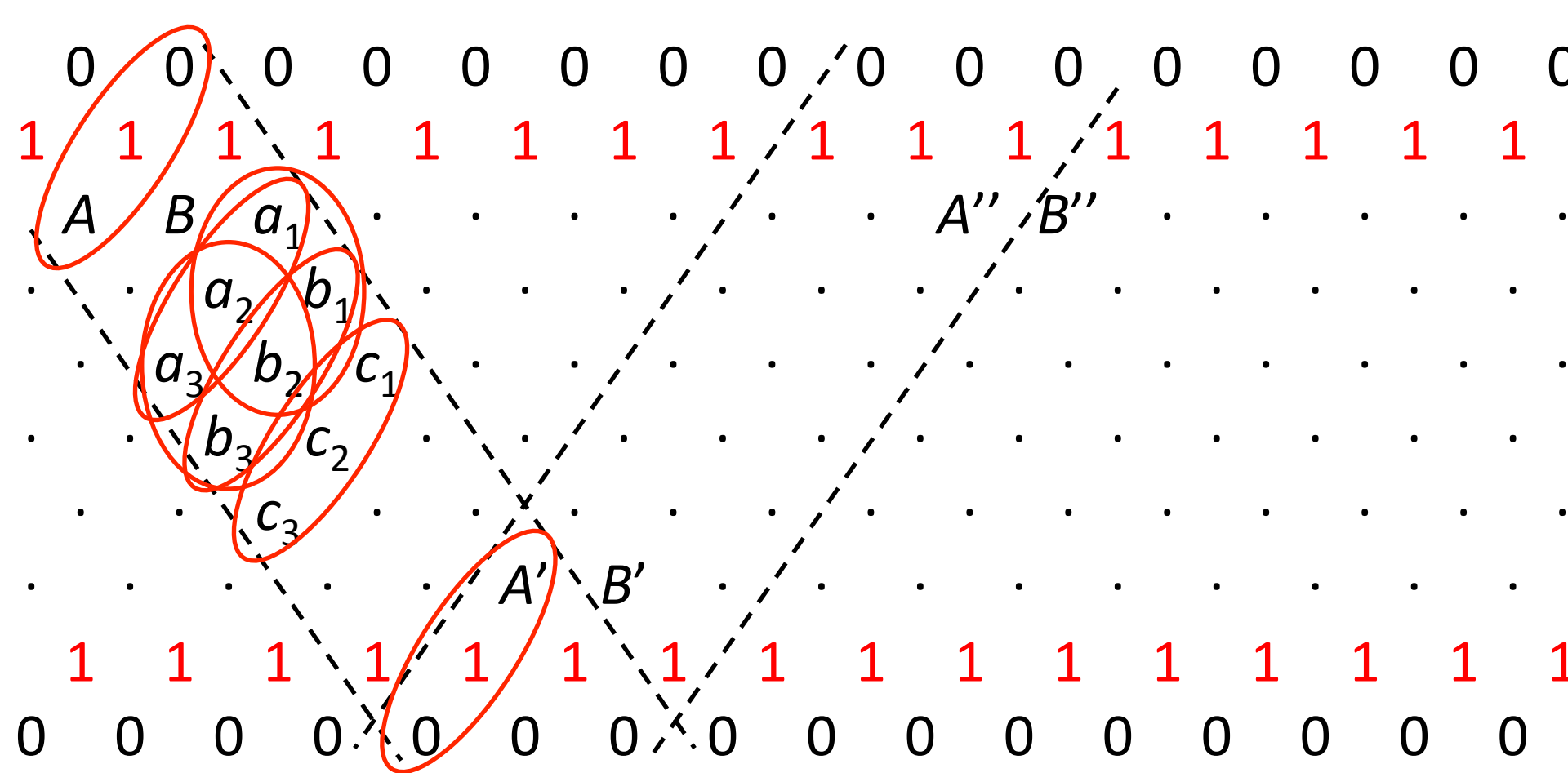
$$bc = ad + 1$$



- Frieze pattern with glide symmetry

$$\begin{matrix} & a & \\ b & & c \\ & d & \end{matrix}$$

$$bc = ad + 1$$



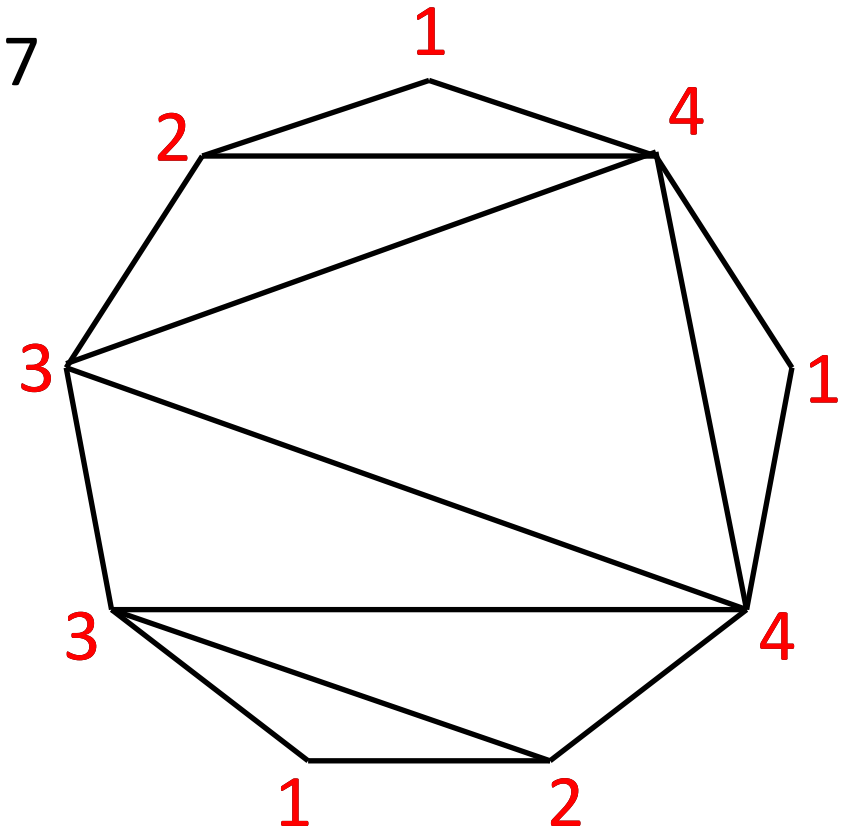
$$a_2 b_1 = a_1 b_2 + 1 \implies (a_1 + a_3) / a_2 = (b_1 + b_3) / b_2$$

$$a_3 b_2 = a_2 b_3 + 1 = (c_1 + c_3) / c_2 \dots$$

$$\therefore A = A' = A'' \dots \quad B = B' = B'' \dots$$

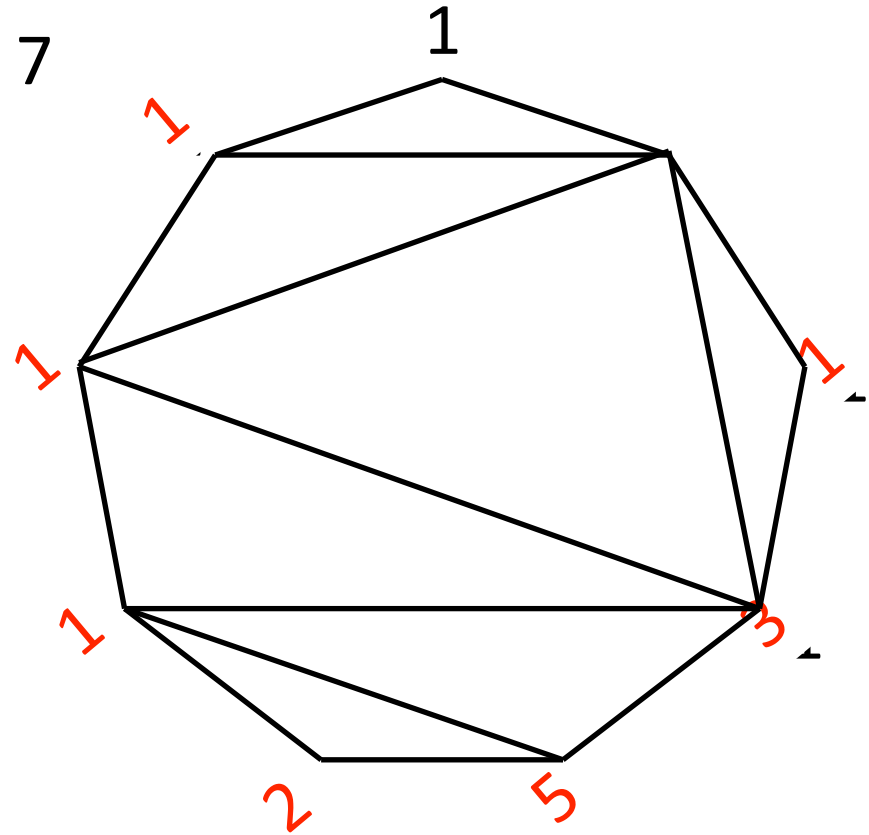
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	3	3	3	7	1	2	8	5	1	3	3	3	7	1	2
2	2	8	5	3	1	5	13	2	2	2	8	5	3	1	5
3	1	5	13	2	2	2	8	5	3	1	5		2	2	2
1	2	8	5	1	3	3	3	7							
2	1	3	3	2	1	4	1	4							
1	1	1	1	1	1	1	1	1							

- Which sequences generate frieze patterns?
- Those which arise from triangulations of a polygon.



1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	4	1	4	2	1	3	3	2	1	4	1	4	2	1	3
1	3	3	3	7	1	2	8	5	1	3	3	3	7	1	2
2	2	8	5	3	1	5	13	2	2	2	8	5	3	1	5
3	1	5	13	2	2	2	8	5	3	1	5		2	2	2
1	2	8	5	1	3	3	3	7							
2	1	3	3	2	1	4	1	4							
1	1	1	1	1	1	1	1								

- Label vertex with 0
- Label neighbours with 1
- Label each vertex with sum of other 2 in triangle



Exercises

- Prove that the rule $(bc=ad+1)$ yields a glide symmetry
- Prove that the pattern is generated by triangulation of a polygon
- This...

Constant values
in each row

1	1	1	1	1	1	1	1	1
$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$
1	1	1	1	1	1	1	1	1

$k^2 = 1 \times 1 + 1$ $k = \sqrt{2}$

No longer
integers

1	1	1	1	1	1	1	1	1
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
1	1	1	1	1	1	1	1	1

$\phi = (\sqrt{5}+1)/2$

What is the
(geometrical)
significance of
these values?

1	1	1	1	1	1	1	1	1
$\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$
2	2	2	2	2	2	2	2	2
$\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$
1	1	1	1	1	1	1	1	1

Can you prove it?

References

- Coxeter, H.S.M and Rigby, J.F. (1961) Frieze patterns, triangulated polygons and dichromatic symmetry, *The Lighter Side of Mathematics*, R.K. Guy and E.Woodrow Eds., 15-27.
- Conway, J.H. And Coxeter, H.S.M. (1973) Triangulated polygons and frieze patterns, *Mathematical Gazette* **57**, 87-94, 175-183.
- Conway, J. (1990), *Quantum*, May, 50-51.
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