Fractran

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A Turing-complete esoteric programming language devised by John Conway.

Turing complete: Can simulate a Turing machine. Conditional branching, loops, etc.

Esoteric programming language: Programming language designed as an experiment, a joke, or a proof of some unusual concept. Examples: INTERCAL, LOLCODE, Whitespace, etc.

John Conway: . . .
Siobhan Roberts, *Genius at Play*, Bloomsbury (2015)

John Conway (1998)

Programs

Program A finite list of positive rational numbers $\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_n}{q_n}\right)$

Input A positive integer m

Output A positive integer

- $i := 1$
- print m
- while $(i \neq n+1)$
  - if mp_i/q_i is an integer
    - $m := mp_i/q_i$
    - print m
    - $i := 1$
  - else
    - $i := i + 1$
- return m

- Try multiplying m by each fraction in the list, until you reach one that gives an integer.
- Set m to this new integer, and start again from the beginning of the list.
- If you make it to the end of the list without getting an integer, stop.
- The output of the program is the final value of m.

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- More generally, if we start with input $m = 2^a 3^b$ then we get

$$2^a 3^b \longrightarrow 2^{a-1} 3^{b+1} \longrightarrow 2^{a-2} 3^{b+2} \longrightarrow \cdots \longrightarrow 3^{a+b}$$

So this program adds two positive integers.

Example: multiplication

The program

$$\left(\frac{455}{33}, \frac{11}{13}, \frac{1}{11}, \frac{3}{7}, \frac{11}{2}, \frac{1}{3}\right)$$

maps $2^a 3^b$ to 5^{ab}, and can therefore be used to multiply two integers.

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For example:

$$36 = 2^2 3^2 \longrightarrow 198 \longrightarrow 2730 \longrightarrow 2310 \longrightarrow 31850 \longrightarrow 26950 \longrightarrow 2450 \longrightarrow 1050$$
$$\longrightarrow 450 \longrightarrow 2475 \longrightarrow 34125 \longrightarrow 28875 \longrightarrow 398125 \longrightarrow 336875$$
$$\longrightarrow 30625 \longrightarrow 13125 \longrightarrow 5625 \longrightarrow 1875 \longrightarrow 625 = 5^4$$

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  - If so, increment v_p and decrement v_q;
  - otherwise, do nothing.
- So if $v_2 > 0$, the instruction $\frac{3}{2}$ increments v_3 and decrements v_2.

Example: Primes

If we start with $m = 2$, the program PrimeGame

$$\left(\frac{17}{91}, \frac{78}{85}, \frac{19}{51}, \frac{23}{38}, \frac{29}{33}, \frac{77}{29}, \frac{95}{23}, \frac{77}{19}, \frac{1}{17}, \frac{11}{13}, \frac{13}{11}, \frac{15}{2}, \frac{1}{7}, \frac{55}{1}\right)$$

generates the sequence

$$\begin{aligned} 2 \longrightarrow 15 \longrightarrow 825 \longrightarrow 725 \longrightarrow 1925 \longrightarrow 2275 \longrightarrow \cdots \\ \longrightarrow 364 \longrightarrow 68 \longrightarrow 4 = \mathbf{2^2} \longrightarrow 30 \longrightarrow 225 \longrightarrow \cdots \\ \longrightarrow 728 \longrightarrow 136 \longrightarrow 8 = \mathbf{2^3} \longrightarrow 60 \longrightarrow 450 \longrightarrow \cdots \\ \longrightarrow 2912 \longrightarrow 544 \longrightarrow 32 = \mathbf{2^5} \longrightarrow 240 \longrightarrow 1800 \longrightarrow \cdots \\ \longrightarrow 11648 \longrightarrow 2176 \longrightarrow 128 = \mathbf{2^7} \longrightarrow 960 \longrightarrow 7200 \longrightarrow \cdots \end{aligned}$$

This program (inefficiently) lists all the primes.

Example: Digits of π

The program PiGame

$$\left(\frac{365}{46},\frac{29}{161},\frac{79}{575},\frac{679}{451},\frac{3159}{413},\frac{83}{407},\frac{473}{371},\frac{638}{355},\frac{434}{335},\frac{89}{235},\frac{17}{209},\frac{79}{122},\frac{31}{183},\frac{41}{115},\frac{517}{89},\frac{111}{83},\frac{305}{79},\frac{23}{73},\frac{73}{71},\frac{61}{67},\frac{37}{61},\frac{19}{59},\frac{89}{57},\frac{41}{53},\frac{833}{47},\frac{53}{43},\frac{86}{41},\frac{13}{38},\frac{23}{37},\frac{67}{31},\frac{71}{29},\frac{83}{19},\frac{475}{17},\frac{59}{13},\frac{41}{291},\frac{1}{7},\frac{1}{11},\frac{1}{1024},\frac{1}{97},\frac{89}{1}\right)$$

maps powers of 2 as follows:

$$1 = 2^0 \longrightarrow 89 \longrightarrow \cdots \longrightarrow 8 = \mathbf{2^3} \longrightarrow \cdots$$

$$2 = 2^1 \longrightarrow 178 \longrightarrow \cdots \longrightarrow 2 = \mathbf{2^1} \longrightarrow \cdots$$

$$4 = 2^2 \longrightarrow 356 \longrightarrow \cdots \longrightarrow 16 = \mathbf{2^4} \longrightarrow \cdots$$

$$8 = 2^3 \longrightarrow 712 \longrightarrow \cdots \longrightarrow 2 = \mathbf{2^1} \longrightarrow \cdots$$

It (very slowly) generates the decimal digits of π.

Joke

My improper fractions helpline is now open 24/7.

— *Rebecca Wilkinson (@debecca)*