On Sequences of Consecutive Semi-Primes

But not even... How odd!

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Plan

1. **Introductions**

2. **The Hunt for $N_{\mathbf{max}}(d)$**

3. **A Nice Result... And New Rabbit Hole.**

What is a Semi-Prime?

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The Warm-Up: $d = 1$ (Consecutive Numbers)

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- $N = 3$: (33, 34, 35)
- $N = 4$: No!... Why not?

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The "Multiple of 4" Argument

- Any 4 consecutive integers must contain one multiple of 4.
- Let this number be $4m = 2 \times 2 \times m$.
- If $m = 1$, the number is 4, (which IS semi-prime!), but looking around 4: $(1, 2, 3, \mathbf{4}, 5, 6, 7 \dots)$. Not all of others are semi-prime. So the chain is $N = 1$.
- If $m > 1$, $4m$ has at least *three* prime factors $(2, 2, \text{and factors of } m)$.
- So $4m$ (for $m > 1$) is **never** semi-prime!

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Result

The longest chain is $N = 3$. So, $N_{\text{max}}(1) = 3$.

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- $N = 4$: Yes! (299, 301, 303, 305)

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Conjecture

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Nope. Here's a counterexample.

- $N = 4$: **299**, 301, 303, 305
- $N = 5$: **213**, 215, 217, 219, 221

Keep Looking...

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The spanner in the works is the next square number... $3 \times 3 = 9$.

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- $(a, a+2, a+4, a+6, a+8, a+10, a+12, a+14, a+16)$

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- This is a **complete set of residues** $\{0, 1, \ldots, 8\}$!
- So: One of them *must* be a multiple of 9.

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The (non-) issue of $m = 1$

- So, any sequence of 9 *must contain the number 9 itself*.
- Let's check the sequence around 9:
- $(\ldots, 3, 5, 7, \mathbf{9}, 11, 13, 15, \ldots)$
- 5, 7, 11, 13 are all PRIME. Which means they aren't semi-prime!

The Result

We're done!

- A chain of 9 (or more) consecutive odd semi-primes cannot exist.
- We found a chain of 8.
- Therefore, the maximum length is 8.

$$N_{\text{max}}(2) = 8$$

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Is this true? I have no idea! But I want to find out!

Thanks!

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